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Application of the Variational-Asymptotical Method to Laminated Composite Plates

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The variational-asymptotical method is a mathematical technique by which the geometrically nonlinear, three-dimensional analysis of laminated plate deformation can be split into a linear, one-dimensional, through-the-thickness analysis and a nonlinear, two-dimensional, plate analysis. The elastic constants used in the plate analysis are obtained in closed form from the through-the-thickness analysis, along with approximate, closed-form three-dimensional distributions of displacement, strain, and stress. The development of such a theory is presented herein for laminated plates in which each lamina exhibits monoclinic symmetry about its own midplane. The resulting theory is termed "neoclassical" because of its simplified treatment of shear deformation and restriction to only the first asymptotical approximation. The stiffness constants for the theory include those of classical laminated plate theory (CPT) plus a 2×2 matrix of transverse shear stiffnesses, the values of which are given in closed form. The analysis is applied to the linear cylindrical bending of laminated plates with arbitrary stacking sequences, and results for various displacement, strain, and stress distributions are compared with those of the exact solution and CPT. The neoclassical theory is shown to exhibit a significant improvement over CPT for the transverse displacement of a variety of laminated plates. Moreover, modest to significant improvements are exhibited in the accuracy of three-dimensional displacement, strain, and stress components.

Introduction

FOR aerospace structures, composite materials provide excellent opportunities for structural simplicity as well as elastic couplings for potential optimization of design criteria. Simple and efficient methods for analyzing plates with anisotropy and nonhomogeneity are still needed, given the rapid changes taking place in manufacturing technology. This paper presents an analysis of laminated plates based on an asymptotical method.

Previous Work

Since 1850, when Kirchhoff formulated his plate theory, numerous papers on plate theory have been published. Although classical plate theory (CPT) is adequate for many engineering applications, it has limitations due to the Kirchhoff hypothesis. As composite materials were introduced, this field witnessed the publication of literally thousands of papers, many of which develop higher order theories to account for nonclassical effects such as transverse shear deformation. For example, see Refs. 1 and 2 and the many references cited therein. There are three classes of plate theories: The most common higher order theories involve the use of through-the-thickness power series approximations for displacements and/or stresses. These theories have the advantage of being straightforward to develop. However, this approach constrains the higher derivatives of these quantities to be continuous, which is not in general correct. Less common are "layerwise" theories, in which the number of unknowns depends on the number of layers in laminated plates, such as in the works of Reddy and co-workers. This approach overcomes the problem with the higher derivatives, but not without adding to the computational burden. Finally, there are the so-called "zig-zag" theories, such as Refs. 3–5, in which the derivatives of the displacement field have the possibility of being discontinuous to allow for appropriate jumps in the strain and stress without the necessity of using displacement variables, the number of which depends on the number of layers. Although there are limited results from these theories in the literature, they have exhibited very good correlation with the exact solution. On the down side, however, these theories rely on functional forms for the displacement field that do not always follow logically from a three-dimensional analysis.

Atılgan and Hodges[6] undertook an approach that is different from any of these, although it seems to have more in common with the last category. The nonlinear kinematics of a general plate were developed in terms of a three-dimensional "warping" displacement superimposed on the arbitrarily large translation and rotation of a typical normal line element. Then the variational-asymptotical method of Berdichevsky[7] was applied to obtain the first two approximations for in- and out-of-plane warping of the line element and the corresponding two-dimensional strain energy functions. Two-dimensional elastic constants were derived analytically, as well as three-dimensional displacement, strain, and stress expressions in terms of the two-dimensional strain measures. The approximate solution for the warping is not a power series, in general, nor does it lead to retention of more unknowns than the Reissner-Mindlin shear deformation theory. The displacement is derived from the three-dimensional analysis, not postulated.

The analysis of Ref. 6 was restricted to laminated plates for which each lamina exhibits monoclinic material symmetry about its middle surface. The authors' first approximation is asymptotically correct for this case and coincides with classical laminated plate theory. However, their second approximation is asymptotically correct only when each element of the reduced stiffness matrix $\bar{Q}$ (see Jones[8]) is constant through the thickness of the entire plate. Although the theory is not asymptotically correct otherwise, it was intended for application to laminated composite plates. No numerical results were presented based on this theory in Ref. 6.

Present Approach

The limitation in Ref. 6 stems from the treatment of the transverse shear strain as a higher order (i.e., a second approximation)

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effect and the presence of a term in the transverse strain energy that is linear in the warping. If this term does not vanish identically, the strain energy cannot be minimized, and the second approximation will not be asymptotically correct. This unwanted term does vanish when the reduced stiffness is constant through the thickness. Otherwise, we expect transverse shear effects from Ref. 6 to be inaccurate.

One of the main reasons for developing the theory of Ref. 6 to the second approximation was to include transverse shear deformation. However, one can also include transverse shear deformation in the first approximation, so that the unwanted term appears in a higher approximation than needed in the final theory and is no cause for concern. This approach would appear to be quite appropriate for typical fiber-reinforced composite materials, which have small shear moduli relative to the extension modulus along the fiber direction. One can and should evaluate the theory a posteriori concerning the order assumed for the transverse shear deformation.

Therefore, this paper proceeds along two lines. First we will present a modified version of the theory of Ref. 6 for laminated plates. In this modified theory, shear deformation is present in the first approximation, and the second approximation is not developed; hence, we refer to it as a neoclassical plate theory (NCPT). Second, we will apply NCPT to the cylindrical bending problems of laminated plates that were solved exactly by Pagano.[9,10] Displacement, strain, and stress distributions through the thickness of bidirectional and shear-coupled laminated plates will be compared with the exact solution. This type of validation is similar to that carried out in Refs. 5 and 11, for example. The results, when compared to the exact solution, will serve to partially quantify the regimes of applicability for NCPT.

Analysis

In this section we will first outline the three-dimensional kinematics of the plate, including its displacement, strain, and stress fields. Second, following closely the work of Atılgan and Hodges[6] we apply the variational-asymptotical method to dimensionally reduce the problem from three to two dimensions. Only the first approximation is developed. Finally, we outline the global analysis and specialize it for the case of linear cylindrical bending.

Three-Dimensional Description

Consider a plate of constant thickness h composed of layers, each of which is homogeneous and possesses monoclinic material symmetry about its midplane. Let us introduce Cartesian coordinates x_i so that x_α denotes lengths along orthogonal straight lines in the midsurface of the undeformed plate, and $x_3 = h\zeta$ is the distance of an arbitrary point to the midsurface in the undeformed plate, where $-1/2 \le \zeta \le 1/2$. Throughout the analysis, Greek indices assume values 1 or 2; Latin indices assume values 1, 2, and 3; and repeated indices are summed over their ranges.

Now, letting $\boldsymbol{b}_i$ denote an orthogonal reference triad along the undeformed plate coordinate lines, one can express the position vector from a fixed point O to an arbitrary point as

$$\hat{\boldsymbol{r}}(x_1, x_2, \zeta) = x_\alpha \boldsymbol{b}_\alpha + h\zeta \boldsymbol{b}_3 = \boldsymbol{r}(x_1, x_2) + h\zeta \boldsymbol{b}_3 \quad (1)$$

The position vector to the midsurface is also the average position of points along the normal line, at a particular value of x_1 and x_2, so that

$$\boldsymbol{r} = \int_{-1/2}^{1/2} \hat{\boldsymbol{r}}\, \mathrm{d}\zeta = \langle \hat{\boldsymbol{r}} \rangle \quad (2)$$

The angle brackets are used throughout the paper to denote the integral through the thickness.

Now we introduce the deformed plate orthonormal triad $\boldsymbol{B}_i(x_1, x_2)$, whose orientation relative to $\boldsymbol{b}_i$ can be specified by an arbitrarily large rotation. The orientation of $\boldsymbol{B}_i$ is coincident with $\boldsymbol{b}_i$ when the plate is undeformed and is determined by the plate deformation.

The position vector from point O to any point in the deformed plate can be represented as

$$\hat{\boldsymbol{R}}(x_1, x_2, \zeta) = \boldsymbol{R}(x_1, x_2) + h\zeta \boldsymbol{B}_3(x_1, x_2) + h w_i(x_1, x_2, \zeta) \boldsymbol{B}_i(x_1, x_2) \quad (3)$$

where $\boldsymbol{R}$ is defined as

$$\boldsymbol{R}(x_1, x_2) = \langle \hat{\boldsymbol{R}}(x_1, x_2, \zeta) \rangle = \boldsymbol{r}(x_1, x_2) + \boldsymbol{u}(x_1, x_2) \quad (4)$$

and where $\boldsymbol{u}$ is the plate displacement vector, defined as the position vector from a point on the undeformed plate midsurface to the corresponding point on the average surface of the deformed plate. A schematic of the plate deformation is shown in Fig. 1.

Since previous descriptions of plate deformation include rigid-body motion, kinematical constraints are needed to make the system determinate. The $\boldsymbol{B}_i$ triad is defined so that

$$\boldsymbol{B}_1 \cdot \boldsymbol{R}_{,2} = \boldsymbol{B}_2 \cdot \boldsymbol{R}_{,1} \quad (5)$$

and $\boldsymbol{B}_3 = \boldsymbol{B}_1 \times \boldsymbol{B}_2$ is parallel to $\langle \zeta \hat{\boldsymbol{R}} \rangle$. These definitions give rise to the same kinematical constraints on the warping as suggested by Hodges et al.[12]:

$$\langle w_i \rangle = 0 \qquad \langle \zeta w_\alpha \rangle = 0 \quad (6)$$

We now turn to the strain field, details of which can be found in Ref. 6. First we arrange the six strain components into a matrix form so that

$$\Gamma = \lfloor \Gamma_e \quad 2\Gamma_s \quad \Gamma_t \rfloor^T \quad (7)$$

where Γ_e includes the extensional and in-plane shearing strains, and Γ_s and Γ_t contain the transverse shear and transverse normal strains, respectively. Thus,

$$\Gamma_e = \lfloor \Gamma_{11} \quad 2\Gamma_{12} \quad \Gamma_{22} \rfloor^T \qquad 2\Gamma_s = \lfloor 2\Gamma_{13} \quad 2\Gamma_{23} \rfloor^T$$
$$\Gamma_t = \Gamma_{33} \quad (8)$$

Two-dimensional (plate) generalized strain measures are given in matrix form by

$$\varepsilon = \lfloor \varepsilon_{11} \quad 2\varepsilon_{12} \quad \varepsilon_{22} \rfloor^T \qquad K = \lfloor K_{11} \quad 2\kappa_{12} \quad K_{22} \rfloor^T$$
$$2\gamma = \lfloor 2\gamma_{13} \quad 2\gamma_{23} \rfloor^T \quad (9)$$

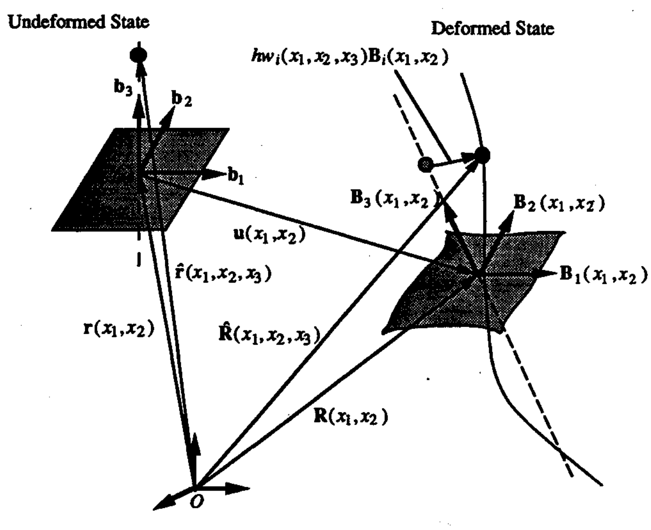

Fig. 1 Schematic of plate deformation.

where ε_{11} and ε_{22} are the plate extensional strain measures, $2\varepsilon_{12}$ is the plate in-plane shear strain measure, $2\gamma_{\alpha 3}$ are the plate transverse shear strain measures, K_{11} and K_{22} are the plate bending measures, and $2\kappa_{12} = K_{12} + K_{21}$ is the plate twisting measure. All of these measures are functions only of x_1 and x_2, and their explicit forms for large deformation are given in Ref. 12.

Denoting the in-plane warping by

$$w_{\|} = \lfloor w_1 \quad w_2 \rfloor^T \tag{10}$$

one can now write the Jaumann strain components following Ref. 6 as

$$\Gamma_e = \varepsilon + \zeta K h + \partial_e^w \qquad 2\Gamma_s = w'_{\|} + 2\gamma + \partial_t w_3$$

$$\Gamma_t = w'_3 \tag{11}$$

where ( )' denotes the derivative with respect to ζ, and we introduce the matrix operators for notational convenience

$$\partial_t = h \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \end{bmatrix} \qquad \partial_e = h \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} \\ 0 & \dfrac{\partial}{\partial x_2} \end{bmatrix} \tag{12}$$

A similar procedure can be followed for the conjugate stresses so that

$$Z_e = \lfloor Z_{11} \quad Z_{12} \quad Z_{22} \rfloor^T \qquad Z_s = \lfloor Z_{13} \quad Z_{23} \rfloor^T$$

$$Z_t = Z_{33} \tag{13}$$

where Z_e contains the extensional and in-plane shear stresses, whereas Z_s and Z_t have the transverse shear and transverse normal stresses. The stress components may then be written in a matrix form as

$$Z = \lfloor Z_e \quad Z_s \quad Z_t \rfloor^T \tag{14}$$

In light of this, the three-dimensional constitutive law can be expressed as

$$\begin{Bmatrix} Z_e \\ Z_s \\ Z_t \end{Bmatrix} = \begin{bmatrix} D_e & D_{es} & D_{et} \\ D_{es}^T & D_s & D_{st} \\ D_{et}^T & D_{st}^T & D_t \end{bmatrix} \begin{Bmatrix} \Gamma_e \\ 2\Gamma_s \\ \Gamma_t \end{Bmatrix} \tag{15}$$

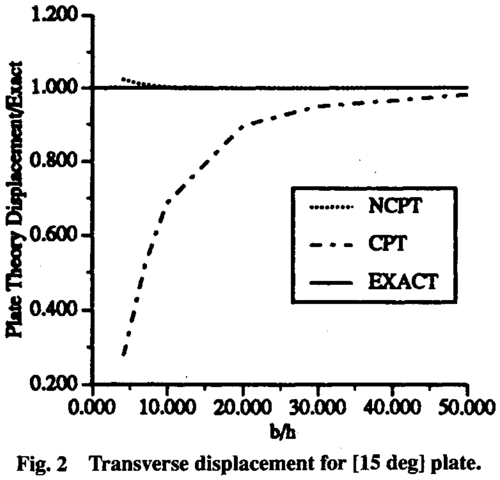

Fig. 2 Transverse displacement for [15 deg] plate.

where D_e, D_{es}, D_{et}, D_s, D_{st}, and D_t are 3×3, 3×2, 3×1, 2×2, 2×1, and 1×1 matrices, respectively. Here, this law is written for directions parallel to plate coordinate axes, which are not in general along the material axis. Therefore, the material constants are the transformed values from the material axis to the plate axis.

The plate strain energy per unit undeformed area can then be written as

$$J = {}^1\!/_2\langle Z^T \Gamma \rangle \tag{16}$$

Following Ref. 6, we decompose the strain energy into two positive definite, quadratic forms. Here we define the extensional strain energy $J_{\|}$ (a function only of Γ_e), and the transverse strain energy $J_{\perp}$ (containing contributions from transverse normal and shear strains) as

$$J_{\|} = \min_{\Gamma_s, \Gamma_t} J \qquad J_{\perp} = J - J_{\|} \tag{17}$$

(Note that the effect of the minimization operation is equivalent to setting transverse shear and normal stress components equal to zero in the strain energy, so that $J_{\|}$ is the strain energy for "plane stress.") When the material fiber direction is oriented parallel to the plane of the plate, such that each lamina exhibits a monoclinic symmetry, D_{es} and D_{st} will vanish. In this case the extensional and transverse energies can be written in terms of the three-dimensional material properties in the following simple form:

$$2J_{\|} = \langle \Gamma_e^T D_{\|} \Gamma_e \rangle$$

$$2J_{\perp} = \langle 2\Gamma_s^T D_s 2\Gamma_s + D_t(\Gamma_t + D_{\perp}\Gamma_e)^2 \rangle \tag{18}$$

where

$$D_{\|} = D_e - D_{et} D_{\perp} \qquad D_{\perp} = D_t^{-1} D_{et}^T \tag{19}$$

This completes the three-dimensional description of the displacement, strain, and stress fields. These three-dimensional fields are not really suitable for plate analysis because of the three-dimensional warping variables w_i. We now turn to elimination of the warping by dimensional reduction.

Dimensional Reduction

In the following sections, we will apply the variational-asymptotical method of Ref. 7 for nonhomogeneous, laminated plates in pursuit of a first approximation of the plate strain energy. Before doing so, however, it is appropriate to discuss the estimation procedure. First we introduce upper bounds on the in-plane, bending, and transverse shear strain measures ε_e, ε_b, and ε_s, respectively, such that

$$\sqrt{\varepsilon^T \varepsilon} \le \varepsilon_e \qquad \frac{h}{2}\sqrt{K^T K} \le \varepsilon_b \qquad \sqrt{2\gamma^T 2\gamma} \le \varepsilon_s \tag{20}$$

For the first approximation we need only to keep terms in the strain field that are of the order of ε where

$$\varepsilon_e + \varepsilon_b + \varepsilon_s \le \varepsilon \tag{21}$$

This implies that we will have strain energy density of the order $\mu\varepsilon^2$ where μ is of the order of the elastic moduli.

To take advantage of the physical aspects of plate deformation, we introduce another small parameter h/ℓ, where ℓ is the smallest constant for which all of the following hold for all possible combinations of α and β:

$$\sqrt{\varepsilon_{,\alpha}^T \varepsilon_{,\beta}} \le \frac{\varepsilon_e}{\ell} \qquad \frac{h}{2}\sqrt{K_{,\alpha}^T K_{,\beta}} \le \frac{\varepsilon_b}{\ell} \qquad \sqrt{2\gamma_{,\alpha}^T 2\gamma_{,\beta}} \le \frac{\varepsilon_s}{\ell} \tag{22}$$

One may think of ℓ as the wavelength of the deformation pattern.

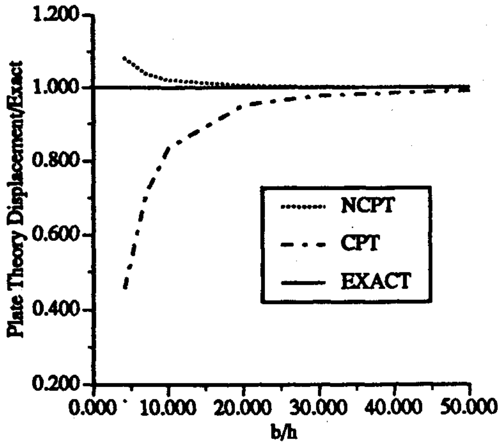

Fig. 3 Transverse displacement for [−15/15 deg] plate.

By consideration of the previous set of estimation parameters, the strain expressions in Eq. (11) can be approximated as

$$\Gamma_e = \varepsilon + \zeta K h \qquad 2\Gamma_s = 2\gamma + w_{\parallel}' \qquad \Gamma_t = w_3' \tag{23}$$

Thus, the warping is not present in extensional energy, and the only part of the strain energy remaining to be minimized is the transverse energy

$$2J_{\perp} = \langle D_t[w_3' + D_{\perp}(\varepsilon + \zeta K h)]^2\rangle + \langle(2\gamma + w_{\parallel}')^T D_s(2\gamma + w_{\parallel}')\rangle \tag{24}$$

The variational-asymptotical method calls for the minimization of this strain energy expression with respect to the warping, subject to the constraints given in Eq. (6), resulting in

$$w_3 = D_{\perp 1}\varepsilon + D_{\perp 2}Kh$$
$$w_{\parallel} = \left[\frac{1}{12}H_s\langle\zeta H_s\rangle^{-1} - I_2\zeta\right]2\gamma \tag{25}$$

in which I_2 is the 2 × 2 identity matrix and

$$D_{\perp 1}' = -D_{\perp} \qquad D_{\perp 2}' = -\zeta D_{\perp} \qquad H_s' = (1-4\zeta^2)D_s^{-1} \tag{26}$$

These expressions for the warping are determined uniquely by imposing the continuity of $D_{\perp\alpha}$ and H_s between the layers together with constraints equation (6) so that

$$\langle D_{\perp\alpha}\rangle = \langle H_s\rangle = 0 \tag{27}$$

for out-of-plane and in-plane warping, respectively. These constraints guarantee that warping functions are continuous. However, strain and stress, which are functions of the derivatives of the warping, are not continuous in general.

The strain energy per unit area of the plate is obtained by substituting the warping from Eq. (25) into the strain energy, yielding

$$J = \tfrac{1}{2}\,[e^T(A\varepsilon + 2BK) + K^T D K + 2\gamma^T G 2\gamma] \tag{28}$$

where

$$A = h\langle D_{\parallel}\rangle \qquad B = h^2\langle\zeta D_{\parallel}\rangle$$
$$D = h^3\langle\zeta^2 D_{\parallel}\rangle \qquad G = h\langle g^T D_s g\rangle \tag{29}$$

and where

$$g = \frac{1}{12}(1-4\zeta^2)D_s^{-1}\langle\zeta H_s\rangle^{-1} \tag{30}$$

A plate theory based on Eq. (30) is geometrically nonlinear because the quantities ε, K, and γ are nonlinear functions of displacements and rotations.[12] Otherwise it looks like a linear theory. Indeed, note that $D_{\parallel}$ corresponds to $\bar{Q}$, the well-known transformed reduced stiffness matrix from classical laminated plate theory.[8] The main difference between a linearized version of our theory and the usual Reissner-Mindlin theory is that our G matrix is a fully populated 2×2 matrix whose elements are determined in closed form in terms of material properties and the geometry of the plate through the thickness.

It is possible to define the force, moment, and transverse shear stress resultants N, M, and Q, respectively, as

$$N = h\langle Z_e\rangle = \left\{\frac{\partial J}{\partial\varepsilon}\right\}^T$$
$$M = h^2\langle\zeta Z_e\rangle = \left\{\frac{\partial J}{\partial k}\right\}^T \tag{31}$$
$$Q = h\langle Z_s\rangle = \left\{\frac{\partial J}{\partial(2\gamma)}\right\}^T$$

Then, based on the strain energy, the plate constitutive law can be expressed as

$$\begin{Bmatrix} N \\ M \\ Q \end{Bmatrix} = \begin{bmatrix} A & B & 0 \\ B^T & D & 0 \\ 0 & 0 & G \end{bmatrix}\begin{Bmatrix} \varepsilon \\ K \\ 2\gamma \end{Bmatrix} \tag{32}$$

If the three-dimensional strains are calculated from the displacement field and, in turn, the transverse normal stress is calculated from the strains via the three-dimensional constitutive law, then the transverse normal stress is zero in this theory. This is, of course, not correct in general. Similarly, we do not expect the transverse strains to be very accurate with the present theory when calculated from the displacement field. There are four reasons for this: 1) Some important contributions to the variation of transverse strains are associated with h/ℓ corrections to the energy.[7] 2) To obtain the transverse strains this way, we need to differentiate the warping with respect to ζ, which will substantially decrease the accuracy of the transverse strains. 3) We have not taken into account any higher approximations of the warping. 4) We have neglected the effects of the applied surface tractions on the warping, the inclusion of which will improve the accuracy of all of the results, including the transverse strains. In some applications it is

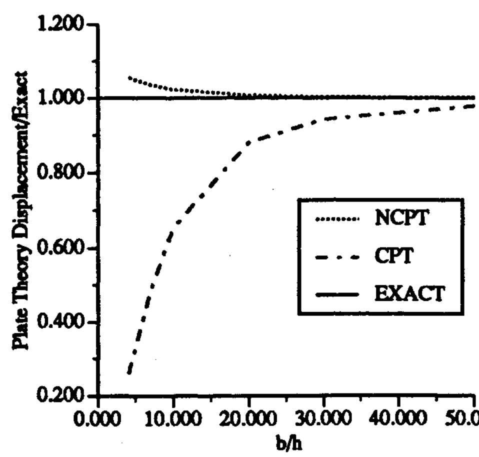

Fig. 4 Transverse displacement for [30/−30 deg]$_{sym}$ plate.

feasible to obtain the transverse strains from integration of the three-dimensional equilibrium equations through the thickness to get the transverse shear and normal stresses, and to apply the three-dimensional constitutive law to get the transverse shear and normal strains. Indeed, in the following cylindrical bending example problem this is done. Note that without any additional work the in-plane strain can be determined with accuracy of a higher order than the transverse strains because of the last term in the first of equations (11).

This concludes the dimensional reduction. The global deformation equations,[12] along with these plate constitutive equations, comprise what we term the neoclassical theory. (Note that the displacement shift mentioned in Refs. 6 and 12 is not necessary with this theory, and thus $2\gamma^* = 2\gamma$.) With the warping known in terms of ε, K, and 2γ, which in turn are known through solution of the global deformation problem, it is now possible to evaluate three-dimensional approximations of displacement, strain, and stress fields. For the purpose of validating the stiffness model and field relations, however, only a specialized version of the global deformation analysis is undertaken here.

Validation

In this section a specialized version of NCPT is developed for the linear cylindrical bending problem. Then NCPT and CPT are applied to a problem for which the exact elasticity solution exists, and the results are compared with the exact solution.

Specialization to Linear Cylindrical Bending for Validation

The global deformation equations that correspond to the previous strain energy function were developed in Ref. 12. Since ε, K, and 2γ are nonlinear functions of the displacement and rotation variables, this theory is applicable to large deformation of plates, and these can now be obtained by solving a specific problem. Here, rather than repeat the entire formulation, we will specialize it for linear, cylindrical bending problems, which we will use below for validation of the dimensional reduction scheme of the previous section.

Linear Plate Equations

Kinematical equations from Ref. 12 for the linear case are given as

$$\varepsilon_{11} = u_{1,1} \qquad K_{11} = \theta_{1,1} \qquad 2\gamma_{13} = \theta_1 + u_{3,1}$$

$$2\varepsilon_{12} = u_{1,2} + u_{2,1} \qquad 2K_{12} = \theta_{1,2} + \theta_{2,1} \qquad 2\gamma_{23} = \theta_2 + u_{3,2} \tag{33}$$

$$\varepsilon_{22} = u_{2,2} \qquad K_{22} = \theta_{2,2x}$$

where $u_i = u \cdot b_i$ and $\theta_\alpha = B_3 \cdot b_\alpha$.

Similarly, equilibrium equations are

$$N_{11,1} + N_{12,2} + f_1 = 0$$

$$N_{12,1} + N_{22,2} + f_2 = 0$$

$$Q_{1,1} + Q_{2,2} + f_3 = 0 \tag{34}$$

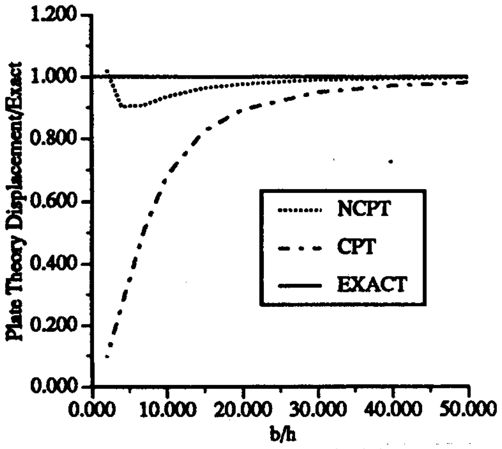

Fig. 5 Transverse displacement for [0/90/0/90 deg] plate.

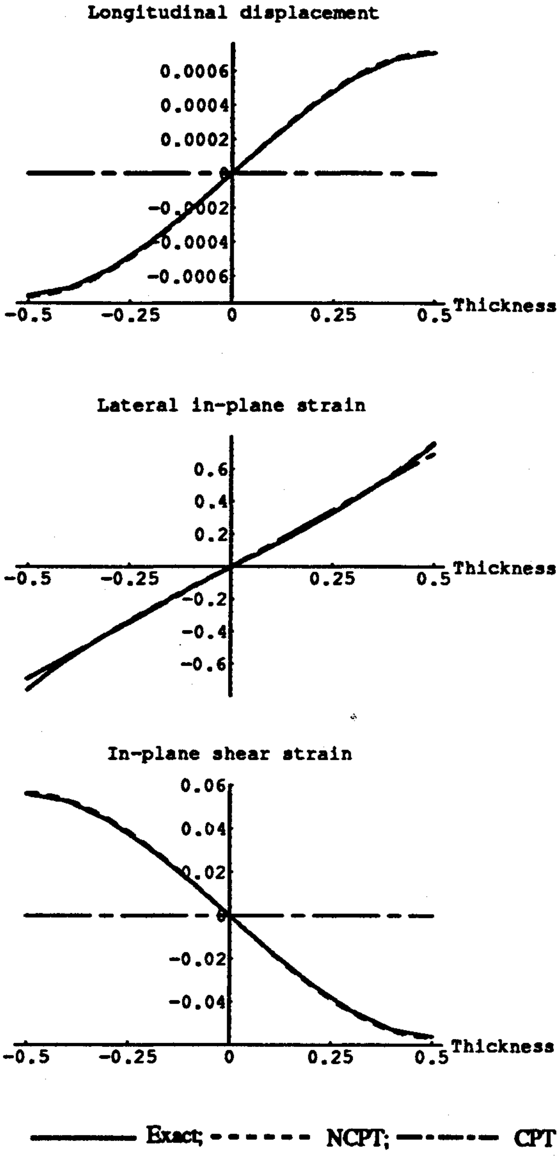

Fig. 6 Distributions of normalized quantities for [15 deg] plate when $b/h = 10$.

$$M_{11,1} + M_{12,2} - Q_1 = 0$$

$$M_{12,1} + M_{22,2} - Q_2 = 0$$

where f_i are the applied loads per unit area of the plate. These along with the constitutive equations, Eq. (32), make up a complete set of 21 equations and 21 unknowns for plate deformation.

Finally, for the purpose of comparing the displacement field with that of linear, three-dimensional elasticity, we need to determine the measure numbers of the displacement field in the bi basis, given by

$$z_i = b_i \cdot (\hat{R} - \hat{r}) \tag{35}$$

With the triad B_i given as

$$B_1 = b_1 + \left(\frac{u_{2,1} - u_{1,2}}{2}\right) b_2 - \theta_1 b_3$$

$$B_2 = \left(\frac{u_{2,1} - u_{1,2}}{2}\right) - b_1 + b_2 - \theta_2 b_3 \tag{36}$$

$$B_3 = \theta_1 b_1 + \theta_2 b_3 + b_3$$

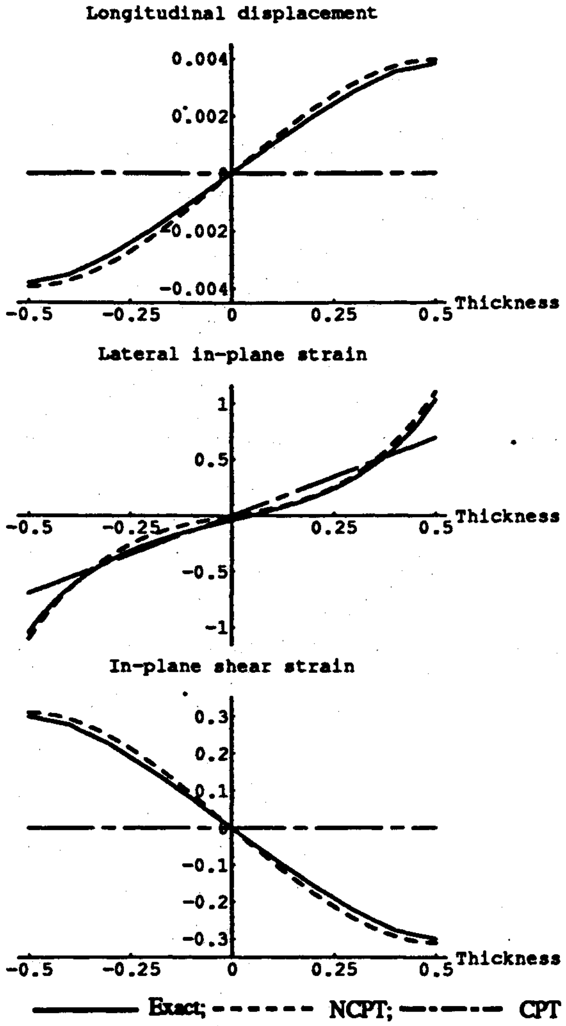

Fig. 7 Distributions of normalized quantities for [15 deg] plate when $b/h = 4$.

the functions z_i can now be expressed in terms of global deformation variables as

$$z_1 = u_1 + h\zeta\theta_1 + hw_1$$

$$z_2 = u_2 + h\zeta\theta_2 + hw_2 \tag{37}$$

$$z_3 = u_3 + hw_3$$

where u_i and θ_α are obtainable from the solution of the plate equations, and the warping wi is known from Eqs. (25) in terms of ϵ, K, and 2γ.

Cylindrical Bending Analysis

Consider a plate of width b along x_1 and infinite length in the x_2 direction. When the plate is subjected to sinusoidal surface loading of the form

$$f_3(x_1) = p_0 \sin\left(\frac{\pi x_1}{b}\right) \tag{38}$$

where we assume the loading f_3 is imposed in the form of equal upper and lower surface tractions for the three-dimensional analysis. Otherwise, one must include warping-dependent terms in the virtual work of the applied loads.[7] This is straightforward but results in a slightly more complex analysis. All derivatives with respect to x_2 are zero, which causes ϵ_{22} and K_{22} to vanish and N_{22} and M_{22} to drop out of the equations, resulting in 17 equations and 17 unknowns. We consider a simply supported plate, the boundary conditions of which are

$$u_3(0) = u_3(b) = 0 \qquad u_{\alpha,1}(0) = u_{\alpha,1}(b) = 0$$

$$\theta_{\alpha,1}(0) = \theta_{\alpha,1}(b) = 0 \tag{39}$$

The solutions for all quantities are of the form of constants times either sin $\pi x_1/b$ or cos $\pi x_1/b$. Details are not given here because of space limitations. However, to recover the three-dimensional fields, one can substitute values obtained for ϵ, K, and 2γ and their derivatives into Eqs. (25), (23), and (15) to get warping, strain, and stress, respectively. Note that all the strain measures can be calculated to order ϵ by using Eq. (23); however, Γ_e can be determined to order $h\epsilon/\ell$ by use of Eq. (11). To avoid inconsistency, however, one should use Eq. (23) for all strain components when calculating the stress components.

Recall from above that the transverse shear and normal stresses and strains need to be handled differently from the others. If they are calculated based on Eqs. (25), (23), and (15) they will not be accurate, because of the need to take a derivative of the warping with respect to ζ. To circumvent this difficulty, we will find the transverse stresses by integration of the three-dimensional equilibrium equations, and the transverse strains can then be found by use of Eq. (15).

Description of Example Problem

The exact three-dimensional elasticity solution for cylindrical bending of angle-ply and cross-ply laminated plates was obtained by Pagano,[9,10] and the results presented herein labelled as "exact" were generated from his equations. The plate in these problems is of width b, of infinite length, and of thickness h; we align our coordinate axes with x_1 along the width ("lateral") and x_2 along the length ("longitudinal"). The material properties are chosen to be

$$E_L = 25 \times 10^6 \text{ psi} = 172.369 \text{ GPa}$$

$$E_T = 10^6 \text{ psi} = 6.895 \text{ GPa}$$

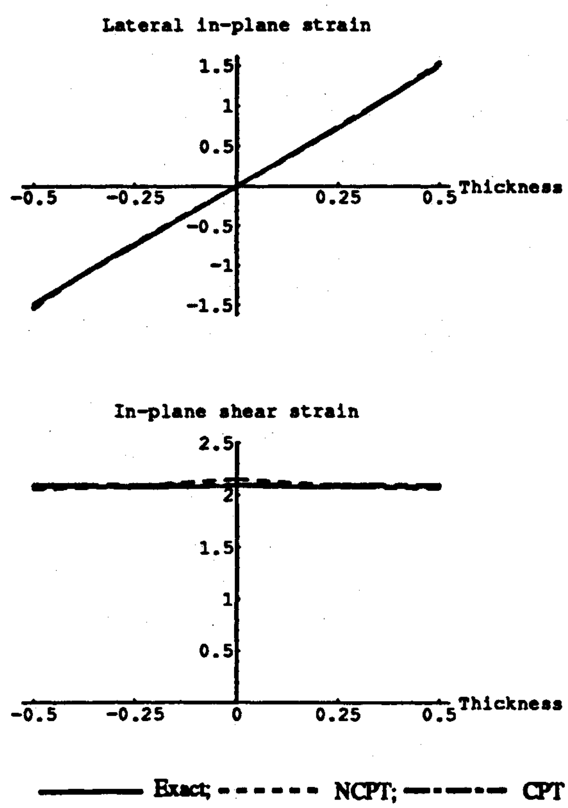

Fig. 8 Distributions of normalized quantities for [−15/15 deg] plate when $b/h = 10$.

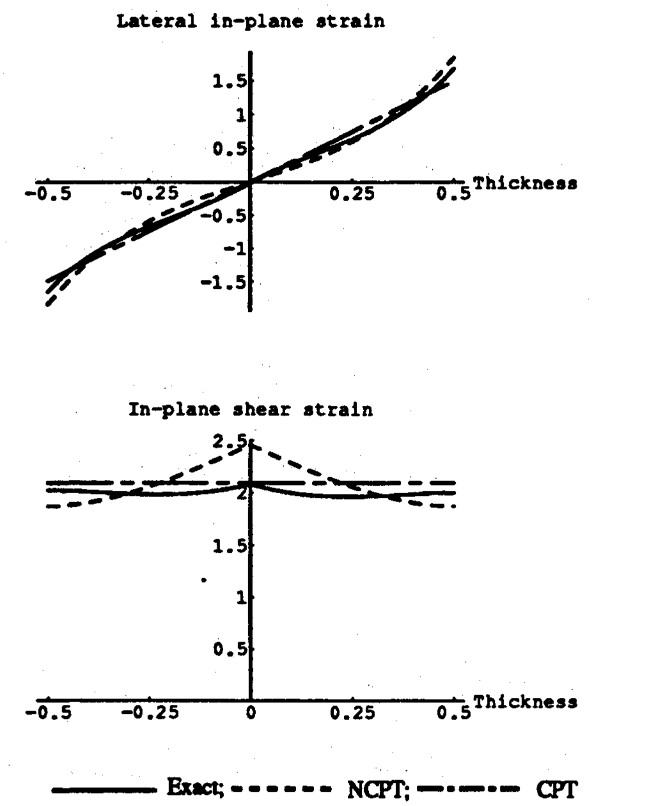

Fig. 9 Distributions of normalized quantities for [−15/15 deg] plate when $b/h = 4$.

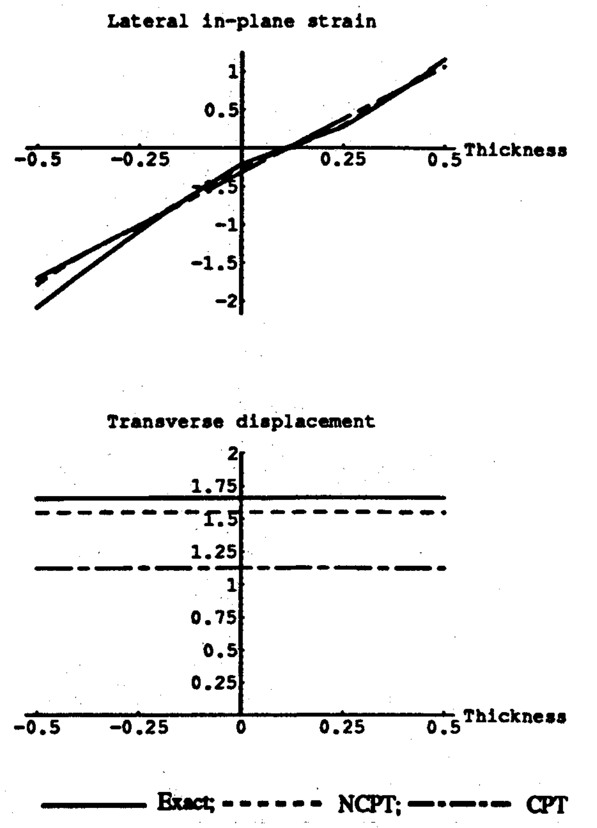

Fig. 10 Distributions of normalized quantities for [0/90/0/90 deg] plate when $b/h = 10$.

$$G_{LT} = 0.5\times 10^6 \text{ psi} = 3.447 \text{ GPa} \tag{40}$$

$$G_{TT} = 0.2\times 10^6 \text{ psi} = 1.379 \text{ GPa}$$

$$\nu_{LT} = \nu_{TT} = 0.25$$

where L signifies the direction parallel to the fibers and T the transverse direction.

The complete solution involves determination of the elastic constants, the solution of the two-dimensional (plate) variables, and the recovery of three-dimensional quantities; details are given below. We will look at the plate transverse displacement as a function of b/h and at three-dimensional quantities obtainable from NCPT, which include the displacement, stress, and strain components. These are compared with results of the exact solution and of CPT for b/h values of 10 and 4, which represent relatively thin and thick plates, respectively. For plotting purposes, the following normalized parameters are used, following Pagano's scheme

$$\bar{z}_\alpha = \frac{E_T h^2 z_\alpha(x_1,\zeta)}{b^3 p_0} \qquad \bar{z}_3 = \frac{100 E_T h^3 z_3(x_1,\zeta)}{b^4 p_0}$$

$$\bar{Z}_{\alpha\beta} = \frac{h^2 Z_{\alpha\beta}(x_1,\zeta)}{b^2 p_0} \qquad \bar{Z}_{\alpha 3} = \frac{Z_{\alpha 3}(x_1,\zeta)}{b p_0}$$

$$\bar{\Gamma}_{ij} = \frac{E_L h^2 \Gamma_{ij}(x_1,\zeta)}{b^2 p_0} \qquad \bar{Z}_{33} = \frac{Z_{33}(x_1,\zeta)}{p_0} \tag{41}$$

The distributions of normalized displacements, strains, and stresses are obtained for laminated plates in which each layer has the same thickness. The following layups are considered: [15 deg], [−15/15 deg], [0/90/0/90 deg], and [30/−30 deg]$_{\text{sym}}$.

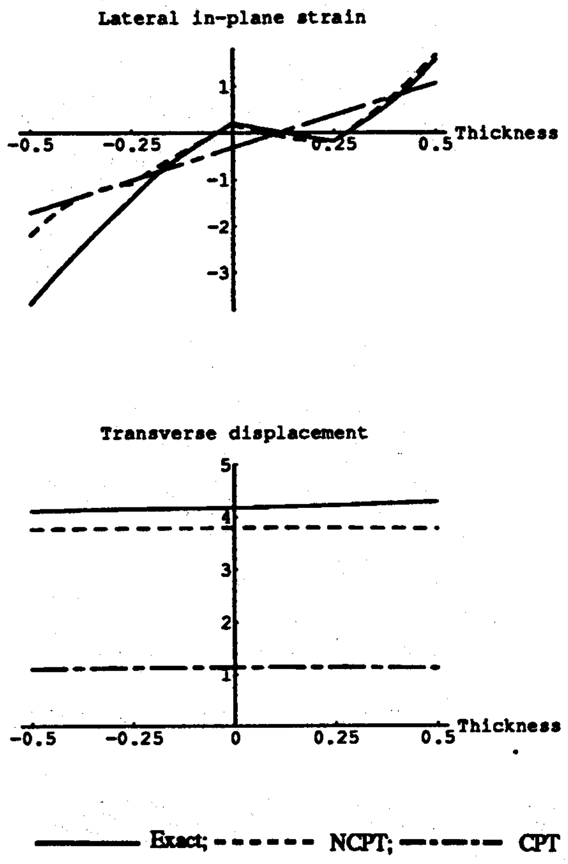

Fig. 11 Distributions of normalized quantities for [0/90/0/90 deg] plate when $b/h = 4$.

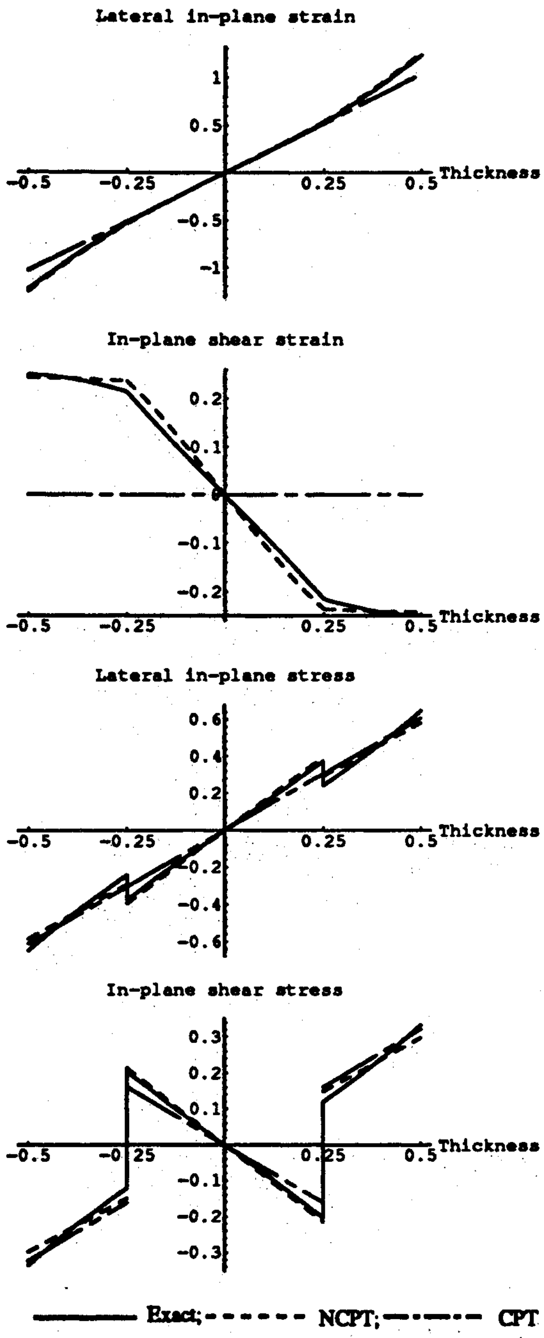

Fig. 12 Distributions of normalized quantities for [30/−30 deg]$_{sym}$ plate when $b/h = 10$.

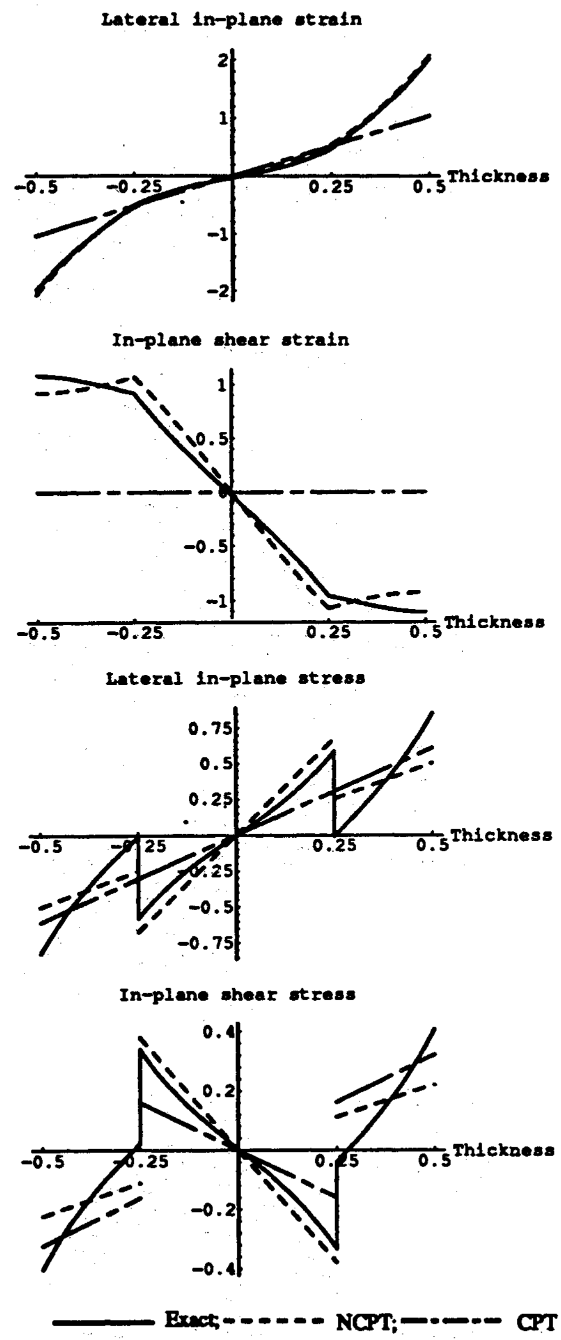

Fig. 13 Distributions of normalized quantities for [30/−30 deg]$_{sym}$ plate when $b/h = 4$.

Results

Results for these plates are presented in Figs. 2–15. Results obtained from application of the theory of Ref. 6 are either very close to those of NCPT or slightly inferior to them. Thus, they are not shown here. The through-the-thickness distributions of in-plane displacements, transverse shear stresses and strains are evaluated at $x_1 = 0$, whereas the distributions of out-of-plane displacement, in-plane stresses and strains, and transverse normal stress and strain are evaluated at $x_1 = b/2$. In all the results following, solid lines represent the exact solution, whereas dashed lines represent the NCPT results. Results from CPT, when distinct from NCPT results, are shown with long and short dashes.

Transverse Displacement

Figures 2–5 show the mid-plane transverse displacement for these four configurations. NCPT and CPT solutions are normalized with the exact solution. The magnitudes of the transverse displacements from CPT are too small. This is because CPT does not account for shear deformation, making it too stiff. NCPT provides a significant improvement over CPT as b/h becomes small, and it remains a good approximation until b/h is down around 10. However, NCPT is stiffer than the exact solution only for the bidirectional case. Otherwise it is slightly more flexible than the exact solution.

Three-Dimensional Quantities

Figures 6–15 show normalized three-dimensional results for NCPT versus those from the exact solution and CPT. Three-dimensional quantities are shown only for b/h values of 4 and 10. We will avoid showing results for which NCPT and CPT are essentially the same. Also, longitudinal in-plane stresses (Z_{22}) are not shown since the curves have the same shape as the corresponding lateral in-plane stresses (Z_{11}). Recall that all transverse shear stresses shown in the figures are obtained from integrating the three-dimensional equilibrium equations, and transverse shear strains are obtained from the stresses via the three-dimensional constitutive equations.

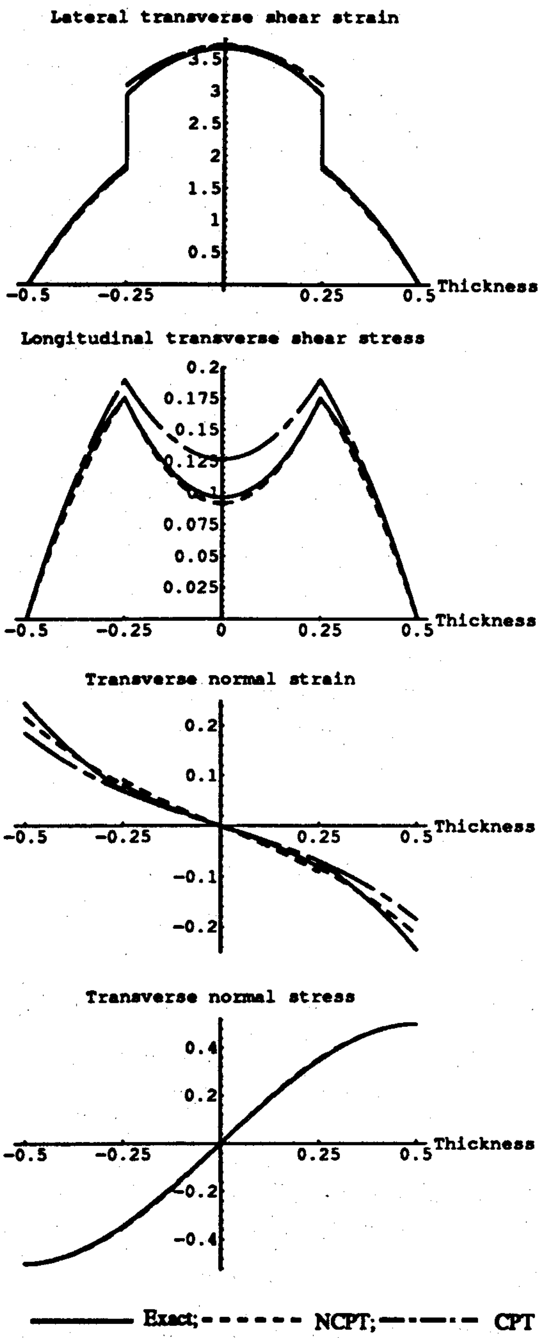

Fig. 14 Distributions of normalized quantities for [30/−30 deg]$_{sym}$ plate when $b/h = 10$.

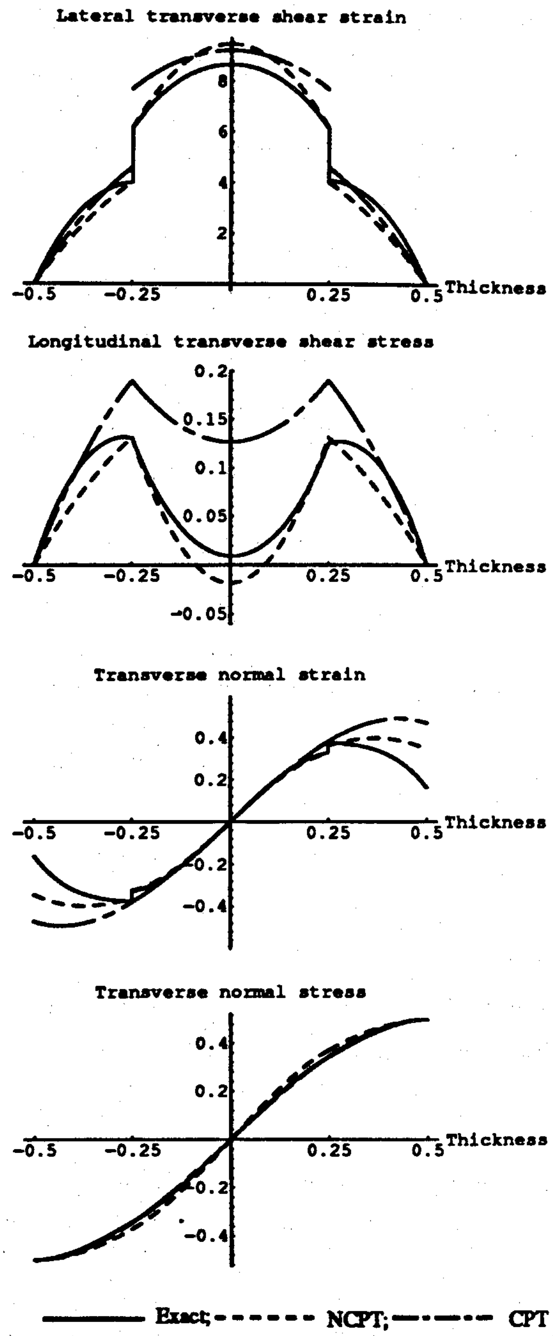

Fig. 15 Distributions of normalized quantities for [30/−30 deg]$_{sym}$ plate when $b/h = 4$.

Figures 6 and 7 show normalized longitudinal in-plane displacement $\bar{z}_2$, lateral in-plane strain $\bar{\Gamma}_{11}$, and in-plane shear strain ($2\bar{\Gamma}_{12}$ for the [15 deg] plate. Figures 8 and 9 show normalized lateral in-plane strain $\bar{\Gamma}_{11}$ and in-plane shear strain $2\bar{\Gamma}_{12}$ for the [−15/15 deg] plate. Note that the longitudinal in-plane displacement has the same form as the mirror image of the in-plane shear strain. This is also true of the lateral in-plane displacement relative to the lateral in-plane strain. We need not show any more in-plane displacement plots for this reason. Even for thick plates NCPT exhibits outstanding agreement with the exact solution. Note the in-plane shear strain and longitudinal displacement from CPT are identically zero while those results from NCPT are close to the exact solution. For all other stress and strain results, NCPT results are the same as those of CPT for this plate.

Figures 10 and 11 show results for normalized lateral in-plane strain $\bar{\Gamma}_{11}$ and transverse displacement $\bar{z}_3$ for the [0/90/0/90 deg] plate. This plate has no shear coupling, and it is evident that the correlation for the lateral in-plane strain is not as good as for the shear-coupled cases shown in Figs. 6–9. The accuracy of the transverse displacement is typical of all the results obtained in this study except that, as noted in the discussion about Figs. 2–5, the absence of shear coupling for this plate results in a stiff model for this case. For all other stress and strain results for this plate, NCPT and CPT exhibit essentially identical behavior.

Figures 12–15 show the results for the [30/−30 deg]$_{sym}$ plate. Figures 12 and 13 show normalized lateral in-plane strain $\bar{\Gamma}_{11}$, in-plane shear strain $2\bar{\Gamma}_{12}$, lateral in-plane stress $\bar{Z}_{11}$, and in-plane shear stress $\bar{Z}_{12}$. Figures 14 and 15 show normalized lateral transverse shear strain $2\bar{\Gamma}_{13}$, longitudinal transverse shear stress $\bar{Z}_{23}$, transverse normal strain $\bar{\Gamma}_{33}$, and transverse normal stress $\bar{Z}_{33}$. This is a symmetric plate with shear coupling, and there are noticeable differences between NCPT results and those of CPT, with NCPT being much closer to the exact solution. For example, even for thick plates NCPT results are very close to the exact solution for the in-plane shear strain which, according to CPT, is zero! We further note that the transverse shear strains are of the same order as the other strain components for this class of plates. It is interesting to note that the transverse normal strain curves from NCPT are continous while the exact solution exhibits slight discontinuities in all these cases.

In many cases the in-plane displacements and strains from NCPT are much closer to the exact solution than those from CPT. The main reason for the accuracy of the NCPT in-plane strains is that Γ_e is calculated to $\mathcal{O}(h\varepsilon/\ell)$ using Eq. (11). Although the in-plane and transverse stresses are essentially identical to those of CPT for the 1- and 2-layer shear-coupled cases and for the four-layer non-shear-coupled case, NCPT performs better than CPT for the four-layer shear-coupled case. In this specific case, although it drops out of CPT for symmetric laminates, the twisting curvature $2\kappa_{12}$ plays an important role in NCPT when applied to this plate. For the four-layer shear-coupled case the transverse strains from NCPT are also better than those from CPT. Overall, for cases which have nonzero shear-coupling terms, NCPT yields better results than for those cases without them. It should be noted that certain of these results quantitatively agree with results from the so-called "first-order zig-zag theory" presented by Ref. 5.

Although NCPT yields results which are the same or an improvement over CPT in many cases, the in-plane shear strain for the 2-layer shear-coupled plate CPT is better. Based on all results obtained to date, we believe that an extension of this theory to a higher approximation still needs to be developed to improve the correlation of this and other three-dimensional field variables with the exact solution. However, the same type of interaction terms as found in Ref. 6 are present in the energy when a higher approximation is attempted. So, extension to a higher approximation requires some additional considerations which will be the object of future studies.

Concluding Remarks

A simple laminated composite plate theory is developed based on the variational-asymptotical method of Ref. 7. The theory is carried out only through the first approximation, including transverse shear deformation. No ad hoc polynomial expansions of displacement or stress through the thickness are involved, nor is a shear correction factor needed. Rather, the appropriate two-dimensional stiffness model for the plate is derived from a three-dimensional theory, and the dimensional reduction process yields closed-form approximations for three-dimensional displacement, strain, and stress.

Although the theory is applicable for large displacements and rotations, and its governing kinematical and equilibrium equations are the nonlinear equations of Ref. 12, only a simple linear validation is carried out herein for cylindrical bending. Results are presented from this theory and compared with the exact solution.

Correlation of results with the exact solution demonstrate that the present "neo-classical" plate theory (NCPT) is superior to classical laminated plate theory for determining the transverse displacement of laminated plates. NCPT also provides improved estimates of three-dimensional variables, such as in-plane strain, stress, and displacement. Three-dimensional results are somewhat better for shear-coupled cases than for the bidirectional case.

Relative to the theory of Ref. 6, NCPT is somewhat simpler, but it has about the same accuracy. The reasons that the theory of Ref. 6 is no more accurate than NCPT, and in some cases inferior to it, are the neglected term of Ref. 6 and the fact that the estimation procedure of Ref. 6 regards transverse shear strain as a higher-order effect. The present results indicate that transverse shear strain is of the same order as the maximum strain for this class of plates.

Additional work needs to be done to extend the range of validity of the first approximation for layered, generally anisotropic plates. More critical is the need to develop the theory to higher approximations. When one attempts to extend the present theory to a higher approximation, terms that are linear in the warping appear in the energy associated with the second approximation which preclude its being minimized, similar to the situation described in Ref. 6.

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